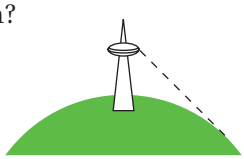
4.4 Modelling With Formulas

The CN Tower, in downtown Toronto, is the world's tallest freestanding structure. It was built in the 1970s as a giant radio antenna. Every year, thousands of tourists enjoy a thrilling ride to the top and a breathtaking view. Have you ever been up the CN Tower? How far can you see?

Investigate

How can you use formulas to solve problems?

Have you ever seen a ship disappear over the horizon? This happens because of the curvature of Earth. The higher you are above the surface of Earth, the farther away the horizon appears. The relationship between how high you are and how far you can see is given by the **formula**

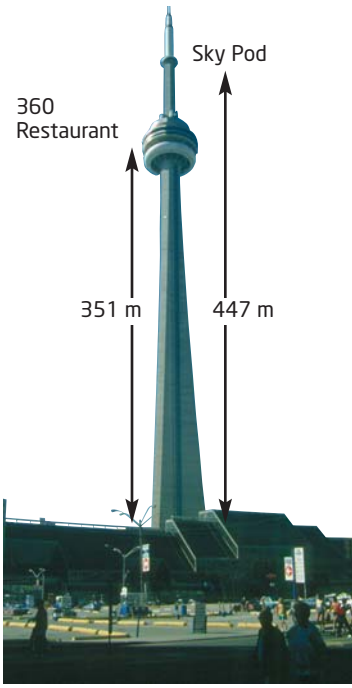


formula

- describes an algebraic relationship between two or more variables

$$d = 2\sqrt{3.2h}$$

where h is your height, in metres, above ground and d is the distance, in kilometres, to the horizon. Use this relationship to answer the following.



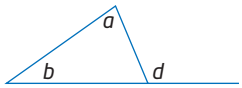
1. How far can you see from each? Round your answers to the nearest kilometre.
   - **a)** the 360 Restaurant
   - **b)** the Sky Pod observation deck
2. **a)** How high would you have to go for the horizon to appear to be 100 km away? Round your answer to the nearest metre.
   - **b)** Explain how you found your answer.
3. **Reflect** Explain how you can use formulas in different ways to solve problems. Think of other formulas that you have used before. Use one of them to illustrate your explanation.

Example 1 Rearrange a Formula in One Step

Making Connections

You will explore geometric relationships in greater depth in Chapter 7: Geometric Relations.

a) In geometry, an exterior angle of a triangle is equal to the sum of the two opposite interior angles:
$d = a + b$



Rearrange this formula to express a in terms of b and d.

b) The circumference of a circle, in terms of its radius, is given by the formula $C = 2\pi r$. Rearrange this formula to isolate r.

c) The area, A, of a square is related to its side length, l, by the formula $A = l^2$. Find the length, to the nearest tenth of a centimetre, of a side of a square with area 32 cm^2.

Solution

a) $d = a + b$

This formula expresses d in terms of a and b. To express a in terms of b and d, subtract b from both sides of the equation.

$d - b = a + b - b$
$d - b = a$
or $a = d - b$

b) $C = 2\pi r$

$\frac{C}{2\pi} = \frac{2\pi r}{2\pi}$

$\frac{C}{2\pi} = r$

or $r = \frac{C}{2\pi}$

This formula has C isolated. To isolate r, I need to divide both sides of the equation by 2π.

c) $A = l^2$

The formula gives A in terms of l. You can also use it to find l if you are given A.

Method 1: Substitute, Then Rearrange

Substitute $A = 32$.

$32 = l^2$

$\sqrt{32} = l$

$5.66 \doteq l$

The square of a number gives 32. To find the number, I need to take the square root. I'll use a calculator.

C 32 √ =

The side length of the square is 5.7 cm, to the nearest tenth of a centimetre.

Technology Tip

Calculators vary. The key sequence may be different on your calculator. Check using a value you know, such as $\sqrt{25} = 5$.

Method 2: Rearrange, Then Substitute

Rearrange the formula to express l in terms of A.

$A = l^2$ Take the square root of both sides.

$\sqrt{A} = l$

or $l = \sqrt{A}$

Substitute $A = 32$.

$l = \sqrt{32}$

$\doteq 5.66$

The side length is 5.7 cm, to the nearest tenth of a centimetre.

These two methods each take about the same amount of time and effort. However, by rearranging the formula before substituting, you get a new formula, $l = \sqrt{A}$, that you can use again, if needed.

Example 2 Rearrange a Formula in More Than One Step

The equation of a line relates y to x, m, and b: $y = mx + b$
Rearrange this equation to express x in terms of y, m, and b.

Making Connections

In Chapter 6: Analyse Linear Relations you will learn the significance of m and b in the $y = mx + b$ form of a linear equation.

Solution

To rearrange a formula in terms of an unknown variable,
- isolate the term that contains the unknown variable
- isolate the unknown variable

Method 1: Use Opposite Operations, Pencil and Paper

$y = mx + b$

$y - b = mx + b - b$ Subtract b from both sides to isolate the term containing x.

$y - b = mx$

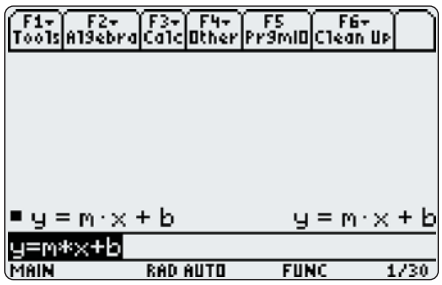
$\frac{y - b}{m} = \frac{mx}{m}$ Divide both sides by m to isolate x.

$\frac{y - b}{m} = x$

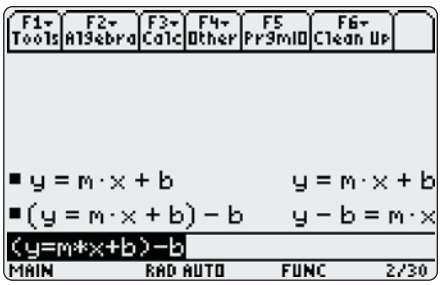
or $x = \frac{y - b}{m}$

Method 2: Use Opposite Operations, Computer Algebra System (CAS)

In the Home screen, type the formula $y = mx + b$. Make sure that you use a multiplication sign between m and x, or the CAS will consider mx as a single variable.



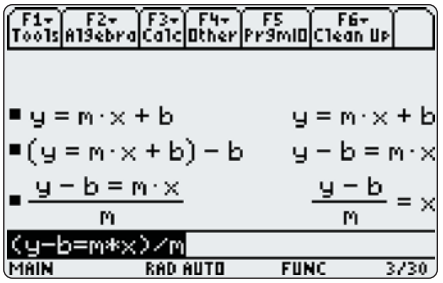
Subtract b to isolate the term containing x.



Divide by m to isolate x.

$$\frac{y - b}{m} = x$$

or $x = \dfrac{y - b}{m}$



Key Concepts

- Formulas can be rearranged to isolate different variables.
- To rearrange a formula in terms of a variable,
  - isolate the term that contains the variable
  - isolate the variable

Communicate Your Understanding

C1 Describe the first step you would take to isolate the variable indicated for each equation.

a) $v = \dfrac{d}{t}$ for d (speed)

b) $y = mx + b$ for m (linear relations)

c) $A = \pi r^2$ for r (area of a circle)

C2 Describe any advantages to rearranging a formula before substituting values.

C3 The formula for the perimeter of a rectangle can be expressed in more than one way. One way is $P = 2(l + w)$. Discuss how you could manipulate this equation to isolate l. Write down the steps and include an explanation of each step. Is there more than one way to do this? Explain.

Practise

For help with question 1, see Example 1.

1. Rearrange each formula to isolate the variable indicated using pencil and paper.

a) $P = 4s$ for s (perimeter of a square)

b) $A = P + I$ for P (investments)

c) $C = 2\pi r$ for r (circumference of a circle)

d) $y = mx + b$ for b (linear relations)

For help with question 2, see Example 2.

2. Rearrange each formula to isolate the variable indicated.

a) $d = mt + b$ for m (distance-time relationships)

b) $P = 2l + 2w$ for w (perimeter of a rectangle)

c) $a = \frac{v}{t}$ for v (acceleration)

d) $v = \frac{d}{t}$ for t (speed)

e) $A = \pi r^2$ for r (area of a circle)

f) $P = I^2R$ for I (electrical power)

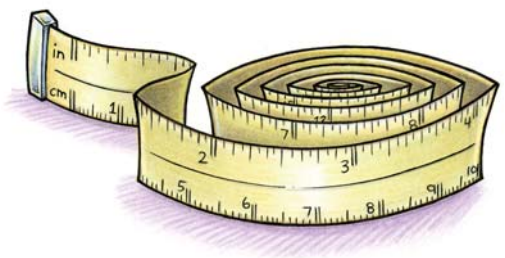
Connect and Apply

3. You can use the formula $C = 2.5I$ to obtain an approximate value for converting a length, I, in inches to a length, C, in centimetres.

a) Use the formula to find the number of centimetres in
- 6 inches
- 3 feet (1 foot = 12 inches)

b) Rearrange the formula to express I in terms of C.

c) How many inches are in
- 75 cm?
- 1 m?



4. Refer to question 3.

a) Plot a graph of $C = 2.5I$, either by hand or using technology such as a graphing calculator or graphing software.

b) Is the graph linear or non-linear? Explain.

c) Use the graph to find
- the number of centimetres in 8 inches
- the number of inches in 35 cm

Making Connections

You learned about linear and non-linear relations in Chapter 2: Relations.

5. Refer to questions 3 and 4. Describe one advantage and one disadvantage of using

a) the equation model **b)** the graphical model

6. Kwok is a hotel manager. His responsibilities include renting rooms for conferences. The hotel charges $250 per day plus $15 per person for the grand ballroom.

a) Create a formula that relates the cost, C, in dollars, of renting the ballroom to the number of people, n.

b) How much should Kwok charge to rent the hall for
- 50 people?
- 100 people?

c) Rearrange your formula to express n in terms of C.

d) How many people could attend a wedding reception if the wedding planners have a budget of
- $4000?
- $2000?

e) In part d), is it better to substitute into the original equation or the rearranged equation? Explain.

f) Is the relationship between cost and number of guests linear or non-linear? Explain how you can tell.

7. **Chapter Problem** As Canadian Superstar nears its finale, the judges have narrowed the competition down to two finalists, Jodie and Quentin. They have one final performance, and the judges will award a score out of 10 for each of the following criteria:
- vocal performance, v
- movement, m
- stage presence, p

The formula $S = 7v + 5m + 3p$ will be used to determine each competitor's overall score, S.

a) After Jodie's performance, the judges awarded her scores of 9 for vocal performance, 7 for movement, and 7 for stage presence. What is Jodie's overall score?

b) It is Quentin's turn. Quentin thinks he can match Jodie on vocal performance, but will likely score only a 6 on movement. Can Quentin win the competition? If so, what is the minimum score he must obtain for stage presence?

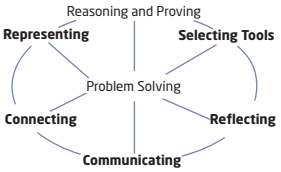
8. The area, A, of a square is related to its perimeter, P, by the formula

$$A = \frac{P^2}{16}$$

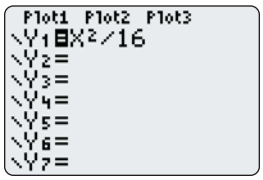
a) Rearrange this formula to express P in terms of A.

b) Find the perimeter of a square with area
- 25 cm^2
- 50 cm^2

9. Refer to question 8.



a) Solve this problem using a graphing calculator or graphing software, by entering the equation shown.



b) Is this a linear or non-linear relation? Explain how you know.

c) Describe two advantages of using an algebraic model to solve this problem.

d) Describe two advantages of using a graphical model to solve this problem.

10. The kinetic energy (energy due to motion) of an object depends on its mass and how fast it is moving:

$$E = \frac{1}{2}mv^2$$

In this formula,

- E is the kinetic energy, in joules
- m is the mass, in kilograms
- v is the speed, in metres per second

The following steps show how the formula can be rearranged to express v in terms of E and m. Copy these steps into your notebook and write a short explanation beside each one. Some hints are provided for you.

Step	*Explanation*
$E = \frac{1}{2}mv^2$	Start with the original formula.
$2E = mv^2$	_____ both sides of the equation by _____.
$\frac{2E}{m} = \frac{mv^2}{m}$	
$\frac{2E}{m} = v^2$	
$\sqrt{\frac{2E}{m}} = \sqrt{v^2}$	Take the _____ _____ of both sides.
$\sqrt{\frac{2E}{m}} = v$	

Did You Know?

The formula in question 10 will remind you of Albert Einstein's famous equation $E = mc^2$. In 1905, while he was studying for his Ph.D. at the University of Zurich in Switzerland, Einstein published several papers which revolutionized scientific ideas. Included in these publications was his discovery that Energy = mass × (speed of light)^2. This property has had huge implications in many fields. In medicine, it is the basis on which modern diagnostic imaging scanners work.

11. Use the formula in question 10 to solve this problem.

Two koala bears, Rocco and Biff, are wrestling in a meadow, when suddenly they notice a nice clump of eucalyptus fall to the ground. Both bears bolt for the tasty treat at the same time. Rocco's mass is 5 kg and he has 4.2 J of kinetic energy. Biff is half a kilogram heavier than Rocco, but has 1 J more of kinetic energy.

a) Who will reach the eucalyptus first? Explain how you know.

b) How much more kinetic energy would the slower bear have to exert for the two bears to reach the eucalyptus at the same time?

12. Sometimes the same formula can have many different forms. $PV = nRT$ is a useful formula in chemistry. It relates the characteristics of a gas:

Variable	Characteristic
P	pressure
V	volume
R	universal gas constant
n	number of moles, or how much gas there is
T	temperature

a) Rearrange this formula to isolate each variable. The first one is done for you as an example:

$PV = nRT$

$\frac{PV}{V} = \frac{nRT}{V}$

$P = \frac{nRT}{V}$ The formula is rearranged to isolate pressure, P.

b) Explain when these forms may be more useful than the form $PV = nRT$.

Extend

Making Connections

You will explore measurement relationships in greater depth in Chapter 8: Measurement and Chapter 9: Optimization.

13. The area, A, of a square is related to its side length, l, by the formula $A = l^2$.

a) Express l in terms of A.

b) Graph both formulas using a graphing calculator or graphing software.

c) How are the graphs similar?

d) How are they different?

14. The volume, V, of a cube is related to its side length, l, by the formula $V = l^3$.

a) Rearrange the formula to express l in terms of V.

b) Graph both formulas and comment on any similarities or differences you see.

15. The distance an accelerating object travels is related to its initial speed, v, its rate of acceleration, a, and time, t:

$$d = vt + \frac{1}{2}at^2$$

a) Rearrange this formula to isolate v.

b) An object travels 30 m while accelerating at a rate of 6 m/s^2 for 3 s. What was its initial speed?

16. Refer to the formula in question 15.

a) Rearrange the formula to isolate a.

b) What acceleration would be required for the object in question 15 part b) to have travelled twice the distance in the same time interval?

17. Math Contest The formula for keyboarding speed (s) is $s = \frac{w - 10e}{t}$, where e is the number of errors, w is the number of words typed, and t is the time, in minutes. Solve the formula for e and find the number of errors made by Saher, who typed 400 words in 5 min, and had a keyboarding speed of 70 words per minute.

18. Math Contest The period (time for one complete swing back and forth) p, in seconds, of a pendulum is related to its length, L, in metres, by the formula $p = 2\pi\sqrt{\frac{L}{g}}$, where $g = 9.8$ m/s^2 is a constant. Solve this formula for L, and find the length needed for the pendulum to have a period of 1 s.

19. Math Contest The escape velocity (speed needed to escape a planet's gravitational field), in metres per second, is given by $v = \sqrt{\frac{2GM}{r}}$.

$M = 5.98 \times 10^{24}$ kg is the mass of the Earth, $G = 0.000\ 000\ 000\ 066\ 73$ (a constant), and r is the radius of the orbit. The average radius of Earth is 6.38×10^6 m.

a) Find the escape velocity for an Earth satellite in kilometres per second.

b) Solve the formula for M.

c) Find the mass of the planet Mars. Mars has a diameter of 6794 km. A Martian satellite requires an escape velocity of 5 km/s.